Durable Products, Time Inconsistency, and Lock-in

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Abstract

Many durable products cannot be used without a contingent consumable product, e.g. printers require ink, iPods require songs, razors require blades, etc. For such products, manufacturers may be able to lock-in consumers by making their products incompatible with consumables that are produced by other firms. We examine the effectiveness of such a strategy in the presence of strategic consumers who anticipate the future prices of both the durable product and the contingent consumable. Under a lock-in strategy, the manufacturer has pricing power over the contingent consumable which she can use to extract additional rents from higher valuation consumers. On the other hand, such pricing power subjects consumers to the possibility of exploitation, i.e. being held-up, after they purchase the durable. We develop a simple model of how consumers derive decreasing marginal utility from the use of a durable and show that the extent to which a manufacturer should pursue a lock-in strategy depends critically upon the level of heterogeneity among consumers. When consumers are homogeneous, then the manufacturer is best off providing access to competitively supplied consumables to eliminate hold-up. On the other hand, as consumers become increasingly heterogeneous, then lock-in is a preferred strategy. In a numerical study, we demonstrate that this insight continues to hold when consumers are worried not only about future consumables prices but also about the manufacturer’s incentive to sell the durable to consumers with lower valuations over time.

1 Introduction

There are many durable products for which manufacturers have devised clever ways of charging consumers based on the amount of use that they derive from the product. Typically, this is done by selling contingent products or services. For example, printers do not print without ink cartridges, commercial aircraft do not fly without replacement parts, etc. Even some sophisticated business application software is nearly impossible to use without expensive consulting and maintenance services. In extreme cases, firms even sell the durables at prices below cost, hoping to make up for this with high margin sales of consumables, a strategy which has been colorfully referred to as, giving away the razor to make money from selling blades.

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There are many ways in which the strategy of locking consumers into contingent products and services can be implemented in practice. When Iomega revolutionized the information storage industry by introducing its Zip drive in 1994, it initially monopolized the market for Zip disks, which were the contingent storage medium that was required to use the Zip drive. But later, these disks were also sold by Fuji, Verbatim, Toshiba, and Maxell. This strategy of initially monopolizing the market for contingent consumables and later allowing competition is not uncommon. In fact, while Gillette typically monopolizes the sales of blades for its most recently introduced razor, it is not uncommon to find generic blade suppliers for older razors. Yet many other firms maintain monopoly control over contingent consumables for extended periods. For example, Abbott Labs remains the sole source of test strips for its FreeStyle glucose monitors; Apple makes it difficult for consumers to download music from web-sites other than iTunes; and many consumer electronics products are compatible with only proprietary peripheral add-ons and accessories.

A lock-in strategy is intuitively appealing as it represents a two-part tariff with respect to consumers in which the price of the durable may be used to extract surplus from consumers and the price of the consumables guides their choices of the quantity of consumption. Indeed, for the case in which there is a single, one-shot, interaction between the manufacturer and a set of perfectly homogeneous consumers, it is relatively obvious that the firm could set the price of consumables to marginal cost and extract the full surplus through the price of the durable. However, there are two main problems with this. First, heterogeneity among consumers interferes with using the price of the durable to extract the full surplus from anyone other than the marginal consumer. Second, consumers are concerned about the extent to which they will be able to derive utility from the durable product after they purchase; for example, consumers of Amazon’s reading device Kindle are concerned about the availability of e-books at reasonable prices (WIRED (2009)). When consumers are locked-in to purchasing a contingent consumable each time they use the durable, they will be concerned about their exposure to being held-up with respect to the price at which consumables are sold. This concern can be explained as follows: Once the manufacturer has sold durables, she may have decreased motivation to continue to provide contingent consumables at a low price, especially if that price is equal to marginal cost. This is problematic for the manufacturer to the extent that consumers’ willingness to pay for the durable is driven by anticipation of the future utility that they will be able to obtain from it. To avoid this problem, one approach that a manufacturer can take is to allow her durable product to be used with contingent consumables other than her own. To the extent that high quality substitute consumables are available from a competitive market, this can eliminate consumers’ fears of being held-up. However, providing access to alternative consumables also means that the manufacturer may need to sacrifice some pricing power and perhaps also some unit-sales in the consumables market. This trade-off between eliminating the hold-up problem with respect to consumers and giving up both sales volume and pricing power in the consumables market is the one upon which we focus our attention.

It is worth pointing out that the hold-up problem that arises with respect to consumers is
due entirely to the fact that, after the manufacturer sells some durables, her incentives for pricing the consumable change, i.e. they are inconsistent over time. Of course, this inconsistency over time of the manufacturer’s incentives is related to the issue recognized by Coase (1972), and later formalized by Bulow (1982), in which consumers’ willingness to pay for a durable product is adversely affected by the anticipation of the manufacturer’s incentive to continue to produce in order to sell to consumers with lower and lower valuations. Because this phenomena arises as the result of the manufacturer’s incentives changing over time, it has often been referred to as time inconsistency.

It is well known that, in the absence of a contingent consumable, a durable good manufacturer can mitigate time inconsistency by leasing her product to consumers. Under a lease, consumers pay a lease fee for the right to use the product for a given period of time but the manufacturer is the residual claimant who owns the product at the end of the lease. By eliminating the manufacturer’s externality, leasing mitigates time inconsistency and allows the manufacturer to earn rents comparable to those in a non-durable good monopoly. Similarly, in the presence of a contingent consumable, if the manufacturer could lease her durable, then consumers would have no reason to anticipate changes in the prices or availability of consumables.

The strategy of locking consumers into a contingent consumable product bears some superficial similarity to a lease since both approaches endow the manufacturer with the ability to charge consumers based on their use of a durable. However, there is a significant distinction: With a lease, consumers pay according to the amount of time for which they have access to the durable, whereas with lock-in, they pay according to their utilization of the durable. Because of this distinction, and because of the fact that lock-in policies are observed frequently in practice, it is of interest to better understand the role of these policies in the presence of strategic consumers. As we will show, if the manufacturer could implement the solution to a mechanism design problem for the consumable in the form of a non-linear pricing policy, then a lock-in strategy would always dominate a policy of allowing consumers to access contingent consumables from a competitive market. However, there are many reasons why we do not often see mechanism design implemented in practice, including the inability of the manufacturer to prevent the resale of consumables. Because of this, as well as the fact that lock-in is often implemented through a linear pricing policy, we devote much of our attention to policies in which the contingent consumable is sold at a fixed price-per-unit.

The rest of the paper is organized as follows. In section 2 we review the literature. In section 3 we develop a model of how consumers derive decreasing marginal utility for a durable product in each period in which they have access to it. After deriving the optimal non-linear pricing policy for the manufacturer, we turn our attention to policies in which the contingent consumable is sold at a constant (endogenous) price-per-unit. We first consider the case in which the manufacturer sells the durable only in the first period, and show that she should prefer lock-in over providing access to competitively supplied consumables only when consumers are sufficiently heterogeneous. In section 4 we extend our analysis to the case in which the manufacturer may produce durables in each of two periods, so that consumers are concerned not only with the future price
of consumables, but also with the manufacturer’s incentive to continue to produce the durable. Numerically, we show that the main insights about the hold-up problem continue to hold in this setting. Finally, we discuss our results and their implications for practice.

2 Literature Review

The strategy that we refer to as lock-in is closely related to a practice known as tying, in which a monopolist in one product (A) market requires its consumers to purchase from it another product (B) in which it does not hold monopolistic power. Whinston (1990) was the first to demonstrate specific conditions, i.e. economies of scale or imperfect competition, under which tying can be beneficial by allowing the firm to make a credible commitment to a higher level of output in the contested market. While Whinston’s analysis does not make any assumptions about the relationship between the two products, later work has focused more specifically on tying two complementary or contingent products. Carlton and Waldman (2002) investigate how tying in the early stages of a product’s life cycle can discourage potential rivals from incurring the fixed costs of entry, while Choi and Stefanadis (2001) obtain similar entry deterrence results when investments in innovation and development are risky.

In contrast to these studies, our examination of lock-in does not assume that there is a one-to-one relationship between the two products. Instead, we focus on a situation in which a consumer’s use of a durable (product A) is linearly related to the amount of a contingent consumable (product B) that he/she is able to obtain. As a consequence, when a monopolist in the market for the durable uses lock-in, she effectively implements a two-part tariff pricing scheme in which the price of the durable is the fixed part and the price of the consumable is the variable part. This introduces a new set of issues, especially when the durable is sold to consumers who expect to use it over time and are strategic in anticipating how the firm will behave with respect to the price of consumables in the future. Because consumer’s anticipation of the monopolist’s incentives may erode their willingness to pay for the durable, we demonstrate that there are conditions under which a lock-in strategy would be dominated by a strategy that welcomes competition in the consumables market.

Several other studies have demonstrated circumstances under which firms can benefit from competition. In studying single product markets, Conner (1995) finds that in the presence of direct network effects, competition from a low-end rival can be beneficial. Sun et al. (2004) extend this analysis and show that strength of network effects plays an important role in determining whether firms should adopt product line extensions, lump sum fee, royalty fee or a free licensing strategy. Another issue that is closely related to the benefits of competition is that of compatibility among the products of different manufacturers of systems of components. Matutues and Regibau (1988) and Economides (1989) recognize that competing firms that manufacture a system of complements choose to make their products compatible to take advantage of the indirect network effects generated from increasing product variety and choice. They show that compatibility
reduces price competition since a reduction in the price of one component leads to increased demand for all systems using that component. Farrell and Katz (2000) study a setting that is a bit closer to ours, and show that a monopolist never loses from independent innovation in a complementary market, and when the effects of innovation can be drastic, the monopolist has a general incentive to cooperate with independent suppliers of the complement. However, in their analysis, the potential benefit from cooperation with independent suppliers is due entirely to the potential for innovation, and they do not consider how the monopolist’s incentives change over time. We show that, even in the absence of the potential for innovation, a monopolist can benefit from facing competition in the market for complements. In addition, we consider the influence of the DGM’s pricing power in the complementary market upon her incentive to price the durable over time.

A durable good monopolist’s strategy of locking consumers into the consumable endows her with pricing power in the market for the complements, at the same time, exposes consumers to a future hold-up problem in the consumables market. As previously mentioned, the issue is closely related to the one first recognized by Coase (1972), in which consumers’ willingness to pay for a durable product is eroded by the manufacturer’s unavoidable temptation to sell to consumers with lower and lower valuations over time. As shown by Bulow (1982), a durable manufacturer can avoid this by leasing her product. However, other studies find that the extent to which leasing is optimal in a durable goods market depends upon several factors, including: the presence of potential entrants (Bucovetsky and Chilton (1986)), competition in the durables market (Desai and Purohit (1999)), depreciation rates of leased and sold products (Desai and Purohit (1998)), interactions with strategic intermediaries (Bhaskaran and Gilbert (2008)), and the availability of complementary products (Bhaskaran and Gilbert (2005)). In the situation in which leasing is not possible, choosing to under-invest in durability and/or employ an inefficient production technology (Bulow (1986)), making public commitment to future wholesale pricing arrangements with intermediaries (Desai et al. (2004)), or the use of intermediaries even in the absence of pre-commitment to wholesale prices to put upward pressure on the price of durables (Arya and Mittendorf (2006)), have been found to address the time inconsistency problem.

When the use that consumers obtain from a durable is tied to a consumable, the DGM’s incentive to exploit existing consumers by raising the price of the consumables once they are locked-in creates a hold-up problem which can be viewed as a form of time inconsistency. In contrast to prior work on durable products, our contribution lies in explicitly recognizing the hold-up (time inconsistency) problem that results from a tying a durable good to a contingent consumable. Although such products are very common, e.g. printers and ink, i-Pod and music downloads, glucose monitoring machine and test strips, etc., we are unaware that anyone has studied the interaction between durables and contingent consumables with a focus on lock-in and its implications for the durable good manufacturers.
3 Model Description

Consider a monopolist manufacturer (M) who produces a durable product from which consumers can obtain utility only by using it in combination with a contingent consumable. In each period for which a consumer has access to the durable, his marginal utility is decreasing in both the amount that he uses the durable and in the quality of the contingent consumable. To represent this, we define the marginal utility function for a consumer of type $a$ as follows:

$$U'(a, s, z) = sa - z$$  \hspace{1cm} (3.1)$$

where $s \in [0, 1]$ is the quality of the consumable, and $z \geq 0$ is the quantity of consumable. This assumption of linearly decreasing marginal utility is similar to the one made by Bhaskaran and Gilbert (2005) in their micro-model of the utility that consumers derive from multiple units of a product that is complementary to a durable. By integrating (3.1), we can derive the following utility for a consumer of type $a$ who consumes quantity $z$ of a consumable of quality $s$:

$$U(a, s, z) = \int_0^z U'(a, s, x)dx = \frac{z(2sa - z)}{2}$$  \hspace{1cm} (3.2)$$

It is worth pointing out that a consumer’s type, $a$, corresponds to her maximum marginal utility for a consumable of quality $s = 1$. We take this type to be uniformly distributed between $[1 - \delta, 1 + \delta]$ across a market of mass equal to one, where $0 \leq \delta \leq 1$, so that the density function for consumer type $a$ is $f(a) = \frac{1}{2\delta}$ for $a \in [1 - \delta, 1 + \delta]$. The parameter $\delta$ allows us to capture the extent of heterogeneity among consumers, where the limiting cases of $\delta \to 0$ and $\delta = 1$ represent a homogeneous consumer population and a highly heterogeneous consumer population respectively.

The argument, $s$, in the consumers’ utility function allows us to consider the possibility that consumers may have access to an alternative supply of consumables. We normalize the quality of the primary consumable that is produced by M to $s = 1$, and denote by $\beta \in [0, 1]$ the relative quality of the alternative consumable. For example, if $\beta = 1$, then the externally supplied consumable is identical to the primary (M’s) consumable, whereas if $\beta = 0$, then the externally supplied consumable provides no utility and effectively does not exist as an alternative. Thus, $\beta = 0$ can be used to represent either the case in which either no externally supplied consumable exists, or the case in which M has chosen to make her durable product incompatible with externally supplied consumables. Obviously, intermediate values of $\beta$ will provide varying amounts of competitive pressure on the price that M can charge for her consumable. To avoid unnecessary complexity, we assume that the market for the externally supplied consumable is perfectly competitive so that it is available at marginal cost.

We assume consumables are perishable in the sense that consumers cannot purchase units in one period and consume them in the future. This is most easily justified for situations in which the contingent consumables are intangible, e.g. the songs or e-books that a consumer may want to
down-load in the future are yet to be created.

Throughout our analysis we normalize the marginal costs of both the durable and the consumable to zero for the purpose of simplifying the exposition. Allowing for positive marginal costs for consumables is straightforward and does not change the fundamental nature of the insights we obtain. Allowing for positive marginal costs for durables is similarly straightforward in the case in which the manufacturer can commit to not producing additional durables after the first period. However, for the case in which he cannot make such a commitment, then positive marginal costs reduce her incentive to produce additional durables beyond the first period, eventually eliminating such incentive altogether. As is common in the durable goods literature, we assume that the performance of the durable does not deteriorate.

Because the durable products that motivate our work are generally not leased to consumers, we do not consider leasing as an option for our manufacturer. Of course, if the manufacturer could lease her durable product, then the hold-up problem with respect to the price of consumables would not exist. In other words, the hold-up problem that is the focus of our research is specific to durable products that are sold, rather than leased, and that require the use of a contingent consumable. On the other hand, there are many reasons why manufacturers cannot lease durable products, including the moral hazard issues that arise when the actions taken by the user of a product are not observable. This may help to explain why leasing is not commonly observed in consumer electronics. In addition, it is worth noting that the products that have motivated our research, e.g. i-Phone, i-Pod, Kindle, personal printers, etc., are overwhelmingly sold rather than leased.

In addition to assuming that the durables are sold, rather than leased, we assume that consumables are sold according to a simple linear pricing mechanism. In practice, there are many obstacles to the use of more sophisticated forms of non-linear pricing, including the difficulty of preventing the resale of “broken” bundles, and consumers’ preference for purchasing consumables as they need them instead of making a single, one-time purchase. Nevertheless, it is of interest to derive the solution to the optimal non-linear pricing policy for the consumable, which can be represented as a mechanism design problem, as a useful benchmark for the simpler, linear pricing policies that are our primary focus.

3.1 Mechanism Design for the Contingent Consumable

In theory, a manufacturer (M) who locks-in her consumers to her own contingent consumable can approach the problem of pricing the consumable in each period as one of mechanism design in which she offers a continuum of quantity and price pairs, which we denote by \( \{q(a), t(a)\} \), so that a consumer of type \( a \) self-selects the quantity and price pair that was designed for him (her). For
our context, the mechanism design problem can be formally stated as:

\[ \text{(MD): } \max_{t(a), q(a)} \int_{1-\delta}^{1+\delta} t(a)f(a) da \]

subject to:

\[ \frac{dV}{da}(a) = q(a) \] (3.3)

\[ V(a) \geq 0 \] (3.4)

\[ q'(a) \geq 0 \] (3.5)

where \( V(a) = U(a, 1, q(a)) - t(a) \) is the information rent received by a consumer of type \( a \in [1 - \delta, 1 + \delta] \). Constraints (3.3) and (3.4) correspond to the incentive compatibility and individual rationality constraints for a consumer of type \( a \). Constraint (3.5) ensures that the second order conditions are satisfied when each consumer self-selects the quantity and price pair, \( \{q(a), t(a)\} \), intended for his type. Define \( MD(\delta) \) to be the value of the optimal solution to problem (MD), and let \( Q^{MD}(\delta) \) be the total fraction of consumers who receive positive quantities of the consumable.

**Lemma 3.1.** The solution to (MD) can be characterized as follows:

i) If \( \delta \leq \frac{1}{3} \): then \( MD(\delta) = \frac{3 - \delta(6 - 7\delta)}{6} \), and all consumers get positive quantities of the consumable, i.e. \( Q^{MD}(\delta) = 1 \). For all \( a \in [1 - \delta, 1 + \delta] \), we have:

\[
q(a) = 2a - (1 + \delta) \quad \text{and} \quad t(a) = \frac{1}{2} \left( 4a(1 + \delta) - 2a^2 - 3(2 - \delta)\delta - 1 \right)
\]

ii) If \( \delta \geq \frac{1}{3} \): then \( MD(\delta) = \frac{(1+\delta)^3}{24\delta} \), and some consumers do not get positive quantities. Specifically, \( Q^{MD}(\delta) = \frac{1 + \delta}{4\delta} \), so that \( q(a) = 0 \) for \( a \leq \frac{1 + \delta}{2} \), while for \( a \geq \frac{1 + \delta}{2} \), we have:

\[
q(a) = 2a - (1 + \delta) \quad \text{and} \quad t(a) = \frac{1}{4} \left( 8a(1 + \delta) - 4a^2 - 3(1 + \delta)^2 \right)
\]

Obviously, if there is nothing to prevent \( M \) from implementing the above solution to the mechanism design problem, then she should certainly do so. Because she can extract the maximum possible surplus from consumers in a given period through sales of the consumable alone, she can give away \( Q^{MD}(\delta) \) durables, as specified in Lemma 3.1, and rely exclusively on the revenues from sales of consumables to locked-in consumers. Because she does not obtain any income from sales of durables, the pricing of consumables will not change from period to period. Thus, the ability to implement the mechanism design solution completely eliminates the hold-up problem with respect to consumers, and \( M \) can make profit \( MD(\delta) \) in each period for which the product is viable.

On the other hand, as we have already mentioned, there are many practical obstacles that might prevent \( M \) from implementing the pricing policy obtained through mechanism design, especially the inability to control third parties from re-selling the consumables. Moreover, even in situations in which such re-selling might be prevented, e.g. when the consumables are distributed
electronically, firms often avoid sophisticated non-linear pricing that is required to implement mechanism design in favor of policies that more closely resemble linear pricing. For example, Apple has steadfastly safeguarded its policy of charging the same price for every song on iTunes. Similarly, while Amazon may charge different prices for individual e-books, it does not attempt to implement any sort of pricing policy in which high volume consumers receive lower per-unit prices. Because such policies are so common in practice, it is of interest to understand when and why a firm might benefit from implementing a lock-in policy in which consumables are priced linearly versus allowing consumers who own the durable to be able to access a consumable that is supplied by a competitive market.

For the remainder of the manuscript, we will focus on situations in which M is restricted to linear pricing policies for the contingent consumable. In Section 3.2, we highlight how such linear pricing policies affect the hold-up problem with respect to consumers by assuming that the manufacturer sells her durable product only in the first period, even though consumers can continue to use it beyond the first period. Under this assumption, we derive analytical results for the conditions under which M is better off locking-in consumers to her own consumable versus when she would be better off allowing them to have access to an alternative consumable of comparable quality that is provided by a competitive market. Subsequently, in Section 3.3 we extend our analysis to the case in which the manufacturer cannot commit to shutting down production of her durable product after the first period.

3.2 Sales of Durables and Consumables with Durable Sales in Period 1 Only

One of the characteristics of the optimal solution to the mechanism design problem is that it allows M to extract the full surplus from the marginal consumer through the price of the consumable alone in each and every period. Consequently, M can rely exclusively upon the income from consumables sales and need not charge anything for the durable. However, once consumables are available to consumers at a constant price per-unit, the only way that M can extract the full surplus from the marginal consumer is by charging consumers a positive price for the use of the durable. Unfortunately, if M sells the durable to consumers who expect to use it for a while, this can create a hold-up problem as consumers anticipate M’s incentive to set the price of the consumable in the future. To demonstrate how this hold-up problem arises, let us consider a situation in which there are exactly two periods. In each period, a consumer of type $a$ has utility $U(a, s, z)$ for $z$ units of a consumable of quality $s$, as defined in (3.2). Let $\rho \leq 1$ be the discount factor applied to period 2. For now, we will assume that the manufacturer produces the durable only in period 1. This could represent a situation in which consumers want to obtain consumables after the durable has gone out of production, but more importantly, it allows us to highlight the main trade-off between a lock-in policy versus one that permits consumers to access consumables produced by a competitive market. The assumption that there are only two periods is not important. It is only important that consumers’ willingness to pay for the durable is affected by their anticipation of the future utility that they will derive from it.
For this setting, the sequence of events is as follows: Prior to the first period, M determines whether to lock consumers into her own contingent consumable or to make her durable compatible with a contingent consumable of quality $\beta$ that is provided by a competitive market at marginal cost (zero). This decision is observed by consumers. In period 1, M announces a per-unit price, denoted by $p_1$, for her own consumable and simultaneously determines an output quantity, denoted by $Q$ for the durable which she sells at the price at which the marginal consumer is indifferent to purchasing the durable$^1$. Equivalently, the manufacturer could announce a price for the durable, and the quantity would be the one at which the marginal consumer is indifferent to purchasing the durable. In period 2, M announces a price for her own consumable, denoted $p_2$, that maximizes her profit from selling consumables to the $Q$ consumers who own the durable. Throughout all of our analysis, we assume that either the same set of consumers are present in both periods or that the distribution of consumer types remains the same and there are no transaction costs in the second-hand market for durables. Either of these two assumptions is sufficient to ensure that the durables are allocated to the consumers with the highest valuation for it.

Before attempting to solve this problem using backward induction, let us do some preliminary analysis of how consumers make choices. After accounting for the price ($p$) of M’s consumable, a consumer of type $a$ will have marginal net utilities of $U'(a, 1, z) - p = a - z - p$ for M’s consumable and $U'(a, \beta, z) = \beta a - z$ for the competitively supplied consumable. It follows that a consumer of type $a$ will have larger marginal net utility for M’s consumable if and only if $p < a(1 - \beta)$, which implies that each consumer will purchase one type of consumable exclusively, either the one provided by M, or the one provided by the competitive market. For whichever type of consumable a consumer chooses, his total net utility will be equal to his total utility, $U(a, s, z)$, less the total amount that he pays to obtain quantity $z$ of the consumable. Thus, he will maximize this net utility by choosing the quantity, $z$, for which $U'(a, 1, z) = p$, for M’s consumable, or $U'(a, \beta, z) = 0$ for the alternative consumable. Defining $z(a, p)$ as the quantity of consumable purchased by a consumer of type $a$ if he holds a durable, we have that:

$$z(a, p) = \begin{cases} 
\beta a & \text{for } a \leq p/(1 - \beta) \\
a - p & \text{for } a \geq p/(1 - \beta) 
\end{cases}$$

(3.6)

To determine the quantity of consumables that M sells, we need to integrate $z(a, p)$ over the consumer types who both hold the durable and prefer M’s consumable to the alternative one. Note that if fraction $Q$ of the market holds the durable, and these durables are allocated to the consumers with the highest marginal utilities, then the marginal consumer will be of type $a_m = \ldots$

$^1$Because we have normalized the mass of the consumer population to one, $Q$ can alternatively be interpreted as the fraction of consumers who hold the durable.
1 + δ(1 − 2Q). The total quantity of consumables sold by M is as follows:

\[
y(Q, p) = \begin{cases} 
\int_{a_m}^{1+\delta} z(a, p) f(a) \, da = Q(1 - p + \delta(1 - Q)) & \text{for } p \leq (1 - \beta)a_m \\
\int_{p \beta / m}^{1+\delta} z(a, p) f(a) \, da = \frac{(p-(1-\beta)(1+\delta))(p(1-2\beta)-(1-\beta)(1+\delta))}{4(1-\beta)^4} & \text{otherwise}
\end{cases}
\] (3.7)

Based on this analysis of how consumers make decisions about purchasing consumables, we can define the problem that M faces in period 2. Let \( \pi_2(Q, p_2) \) be M’s profits in period 2 given that \( Q \) consumers hold a durable. Under the assumption that M sells no additional durables, then her profits can be expressed as:

\[
\pi_2(Q, p_2) = p_2 y(Q, p_2)
\] (3.8)

The above profit function is unimodal in \( p_2 \), and the conditionally optimal price of the consumable in period 2 can be characterized as follows:

**Lemma 3.2.** There exists a threshold, \( \bar{Q} = \frac{\beta(1+\delta)}{\delta(4\beta - 1 + \sqrt{1-2\beta+4\beta^2})} \), such that the conditionally optimal consumable price is:

\[
p_2^* = \begin{cases} 
\min \left\{ (1-\beta)(1+\delta(1-2Q)), \frac{1+\delta(1-Q)}{2} \right\} & \text{for } Q \leq \bar{Q} \\
\frac{(1-\beta)(1+\delta)}{2(1-\beta)+\sqrt{1-2\beta+4\beta^2}} & \text{for } Q > \bar{Q}
\end{cases}
\]

such that, when \( Q \leq \bar{Q} \), M prices her consumable low enough to attract purchases from all consumers who hold the durable. Otherwise, M’s consumables price causes some consumers who hold the durable to prefer to purchase the alternative consumable that is of quality \( \beta \).

The existence of the threshold, \( \bar{Q} \) can be explained as follows: As the quantity of durables that are in use increases, the maximum marginal utility of the marginal consumer \( (a_m = 1 + \delta(1 - 2Q)) \) becomes lower and lower, making M less and less willing to price her consumable sufficiently low to attract purchases from this marginal consumer.

**Corollary 3.3.** \( \bar{Q} \) is decreasing in \( \beta \) and in \( \delta \), while \( p_2^* \) is non-increasing in \( \beta \). \( p_2^* \) is increasing in \( \delta \) for both \( Q < \max \left\{ \frac{1}{2\gamma} \frac{(2\beta-1)(1+\delta)}{(4\beta-1)\delta} \right\} \) and for \( Q > \bar{Q} \). However, if \( \max \left\{ \frac{1}{2\gamma} \frac{(2\beta-1)(1+\delta)}{(4\beta-1)\delta} \right\} < \bar{Q} \), then \( p_2^* \) is decreasing in \( \delta \) for \( Q \in \left( \max \left\{ \frac{1}{2\gamma} \frac{(2\beta-1)(1+\delta)}{(4\beta-1)\delta} \right\}, \bar{Q} \right) \).

As the quality, \( \beta \), of the alternative consumable increases, it begins to put downward pressure on the price that M offers, but at the same time, it makes M increasingly willing to allow the lowest valuation consumers to purchase the alternative consumable. To understand how the heterogeneity parameter, \( \delta \), affects M’s pricing of her consumable, it is useful to consider the average value of \( a \), i.e. the maximum marginal utility, among all those consumers who hold the durable. This
average maximum marginal utility can be calculated as:

\[
\frac{(1 + \delta) + (1 + \delta(1 - 2Q))}{2} = 1 + \delta(1 - Q)
\]

This can be interpreted as the average intercept of the potential consumers' individual linear demand functions, and it is clearly increasing in \(\delta\) for all \(Q \in (0, 1)\). Thus, as \(\delta\) increases, for any \(Q < 1\), the average value of \(a\) among consumers holding the durable increases, and this pushes M toward a higher consumable price. At the same time, an increase in \(\delta\) also increases the gap between the maximum marginal utility for the highest type consumer, \((1 + \delta)\), and that of the marginal consumer, \(a_m = 1 + \delta(1 - 2Q)\), and consequently M becomes increasingly willing to allow these lowest type consumers to purchase the alternative consumable. The one situation in which \(p_2^*\) is increasing in \(\delta\) is when both \(p_2^* = (1 - \beta)a_m\) and \(Q > \frac{1}{2}\). The first of these conditions, \(p_2^* = (1 - \beta)a_m\), occurs when when the optimal price is just low enough to make the marginal consumer indifferent between M’s consumable and the substitute. Note that it corresponds to the one point at which M’s profit is not continuously differentiable, so there can be a range of \(Q\) for which the \(p_2^*\) corresponds exactly to this point. The second condition, \(Q > \frac{1}{2}\), implies that an increase in \(\delta\) decreases \(a_m = 1 + \delta(1 - 2Q)\) and subsequently decreases \(p_2^* = (1 - \beta)a_m\). Intuitively, we can think of this in terms of M’s reluctance to give up the marginal consumer to the substitute consumable by keeping her price just low enough.

We can now turn our attention to period 1, when M determines both the quantity, \(Q\), of durables, and a price, \(p_1\), for the consumable. We first need to determine the price at which M will be able to sell quantity, \(Q\), durables given that consumers are strategic and anticipate the amount of utility that they will be able to derive from the durable in both periods. To do this, we will use a concept called the implicit rental price, which represents the maximum rental fee at which a given quantity of durables could be rented to consumers for a single period of use. Let us define \(r(\beta, Q, p)\) to be the implicit rental price when \(Q\) consumers hold the durable, the price of M’s consumable is \(p\), and an alternative consumable of quality \(\beta\) is available at price zero. In the context of our model, the implicit rental price is the total utility that the marginal consumer of the durable will obtain, net of the price of the consumables. Recall that the type of the marginal consumer is \(a_m = 1 + \delta(1 - 2Q)\). Specifically:

\[
r(\beta, Q, p) = \begin{cases} 
U(a_m, 1, z(a_m, p)) - pz(a_m, p) & \text{for } p \leq a_m/(1 - \beta) \\
U(a_m, \beta, z(a_m, p)) & \text{for } p \geq a_m/(1 - \beta) 
\end{cases}
\]

After substituting the utility function defined in (3.2), and substituting (3.6) for \(z(a, p)\), the above can be expressed as:

\[
r(\beta, Q, p) = \begin{cases} 
(1 + \delta(1 - 2Q) - p)^2 /2 & \text{for } p \leq a_m/(1 - \beta) \\
(1 + \delta(1 - 2Q))^2 \beta^2 /2 & \text{for } p \geq a_m/(1 - \beta) 
\end{cases}
\]
Using this implicit rental price, we can now express the market clearing price for the durable in period 1 as the following:

$$r(\beta, Q, p_1) + \rho r(\beta, Q, p_2^*)$$

(3.10)

where \(p_2^*\) depends upon \(Q\) as described in Lemma 3.2. The above selling price for the durable represents the total discounted utility that the marginal consumer will receive from using the durable net of the price that he/she pays to obtain the consumables. From the expression that we obtained for the implicit rental price in (3.9), we have the following:

**Lemma 3.4.** The price at which a given quantity, \(Q\), of durables can be sold in period 1 is non-decreasing in the quality, \(\beta\), of the alternative consumable that is supplied by a competitive market.

This result can be obtained by substituting (3.9) into (3.10) and using the result from Corollary 3.3 that \(p_2^*\) is non-increasing in \(\beta\). Note that this confirms the intuition that the availability of a high-quality consumable that is supplied by a competitive market can help to mitigate the hold-up problem that results from strategic consumers’ anticipation of M’s incentive to exploit them in the future with high consumables prices. On the other hand, the availability of such a high quality alternative consumable can interfere with M’s ability to use the price of her own consumable as a means of extracting additional surplus from the high valuation consumers. We can now express M’s profit function in period 1 as:

$$\pi_1(\beta, Q, p_1) = Q (r(\beta, Q, p_1) + \rho r(\beta, Q, p_2^*)) + p_1 y(Q, p_1) + \rho \pi_2(Q, p_2^*)$$

(3.11)

In order to provide clear intuition about the conditions under which M can benefit from providing access to an alternative consumable, let us consider and compare two special cases of the problem, \(\beta = 0\), which represents the case in which consumers do not have access to any consumable other than the one supplied by M, and \(\beta = 1\), which represents the case in which consumers have access to an alternative consumable that is of quality identical to that of M’s consumable.

For the special case of \(\beta = 1\), consumers effectively have free access to consumables of the same quality as those supplied by M. Obviously, this eliminates M’s ability to obtain any income from consumables sales, so that she must rely entirely upon the income that she receives from durables. By substituting \(\beta = 1\) and \(p = 0\) into (3.6), we can see that \(z(a, 0) = a\) when \(\beta = 1\), so that each consumer who has a durable obtains an efficient quantity of consumables. By substituting into (3.9), we can see that the implicit rental price in each period will be the following function of \(Q\): \(r(1, Q, 0) = \frac{1}{2} (1 + \delta(1 - 2Q))^2\). Under our current assumption that M sells durables in period 1 only, she will not earn any income in period 2 since \(\beta = 1\) forecloses her sales of consumables, and her total profit can be expressed as:

$$\pi_1(1, Q, 0) = Q (r(1, Q, 0) + \rho r(1, Q, 0)) = Q \frac{(1 + \rho)}{2} (1 + \delta(1 - 2Q))^2$$

(3.12)

**Lemma 3.5.** For the case in which M sells durables in period 1 only and \(\beta = 1\), her optimal quantity of
durables and corresponding profit will be:

\[
Q^{\beta_1} = \begin{cases} 
1 & \text{for } \delta \leq 1/5 \\
\frac{1+\delta}{6\delta} & \text{for } \delta \geq 1/5
\end{cases}
\]

\[
\pi_1^{\beta_1} = \begin{cases} 
\frac{(1+\rho)(1-\delta)^2}{2} & \text{for } \delta \leq 1/5 \\
\frac{(1+\rho)(1+\delta)^3}{27\delta} & \text{for } \delta \leq 1/5
\end{cases}
\]

Note that when \( \delta \leq 1/5 \), consumers are sufficiently homogeneous that M sells the durable to all of them. Only as \( \delta \) increases above this threshold does M begin to ration the durable to only those consumers who are of sufficiently high types.

For the special case of \( \beta = 0 \), consumers do not have access to any consumable other than the one supplied by M, so that M has complete flexibility with respect to the price that she sets for consumables in period 2. By substituting (3.7) into (3.8) and evaluating at \( \beta = 0 \), M’s second period profit function can be expressed as:

\[
\pi_2(Q, p_2) = \begin{cases} 
p_2Q(1 - p_2 + \delta(1 - Q)) & \text{for } p_2 \leq a_m \\
p_2(1 - p_2 + \delta)^2 / 4\delta & \text{otherwise}
\end{cases}
\]

By applying the results from Lemma 3.2, we have that, when M faces no competition from an alternative consumable, the price that she will set for her own consumable in period 2 will be:

\[
p_2^{\beta_0} = \begin{cases} 
(1 + \delta(1 - Q)) / 2 & \text{for } Q \leq (1 + \delta) / 3\delta \\
(1 + \delta) / 3 & \text{otherwise}
\end{cases}
\]

which corresponds to conditionally optimal second period profits of:

\[
\pi_2^{\beta_0}(Q) = \begin{cases} 
Q(1 + \delta(1 - \delta)) / 4 & \text{for } Q \leq (1 + \delta) / 3\delta \\
(1 + \delta)^3 / 27\delta & \text{otherwise}
\end{cases}
\]

Note that when \( Q > (1 + \delta) / 3\delta \), M sets the consumable price to be \( p_2^* > a_m \), i.e. she prices the consumable above the maximum marginal utility of the marginal consumer, which implies that not all consumers who hold the durable will derive any value from it, and this will drive the implicit rental price for period 2 to zero.

In the first period, M determines both \( p_1 \) and \( Q \) to maximize \( \pi_1(\beta, Q, p_1) \), as defined in (3.11), with \( \beta = 0 \) and \( \pi_2(Q, p_2^0) = \pi_2^{\beta_0}(Q) \). The solution to this problem can be characterized as follows:

**Lemma 3.6.** For the case in which M sells durables in period 1 only and \( \beta = 0 \), her conditionally optimal consumables price for any \( Q \) is \( p_1^{\beta_0} = Q\delta \). Her optimal quantity of durables and the corresponding profit can be characterized as follows:

\[
Q^{\beta_0} = \begin{cases} 
1 & \text{for } \delta \leq \frac{4+3\rho}{8(2+\rho)} \\
\frac{(1+\delta)(4+3\rho)}{\delta(20+11\rho)} & \text{otherwise}
\end{cases}
\]

\[
\pi_1^{\beta_0} = \begin{cases} 
\frac{4+3\rho-4(1-\delta)(2+\rho)}{8} & \text{for } \delta \leq \frac{4+3\rho}{8(2+\rho)} \\
\frac{(1+\delta)^3(2+\rho)(4+3\rho)^2}{2\delta(20+11\rho)^2} & \text{otherwise}
\end{cases}
\]
By comparing the results in Lemmas 3.5 and 3.6, we have that:

**Corollary 3.7.** For any values of the parameters $\delta$ and $\rho$, we have $Q^{\beta_0} \geq Q^{\beta_1}$, i.e. competition from a competitively supplied consumable that is of quality comparable to her own causes $M$ to produce fewer durables than when she faces no such competition.

This corollary highlights the fact that, when $M$’s consumers can obtain an alternative consumable from a competitive market, $M$ is left with only one lever for extracting consumer surplus: the quantity / price of her durable. Although this extracts the full surplus from the marginal consumer, it leaves a large amount of surplus in the hands of higher valuation consumers, particularly when $\delta$ is large and consumers vary substantially in their marginal valuations. Consequently, she is relatively reluctant to sell to consumers of low types. In contrast, when $\beta = 0$ so that her consumers do not have access to any consumable other than her own, $M$ can use the price of the consumable to extract some of the additional surplus that is obtained by the higher type consumers. It is easy to confirm from the result in Lemma 3.6 that $p_t^{\beta_0}$ is increasing in $\delta$. Because higher consumables prices result in less efficient quantities of consumables being allocated to consumers of all types, this implies that, as consumers become increasingly heterogeneous, $M$ becomes more willing to accept less efficient allocations of her consumable in return for extracting additional surplus from the highest valuation consumers. In addition, because the the price of the consumable allows $M$ to extract more of the surplus from the highest valuation consumers, it also increases her willingness to put durables in the hands of low type consumers. We can now characterize the conditions under which $M$ prefers to lock-in consumers to her own consumable:

**Proposition 3.8.** For the case in which $M$ sells durables in period 1 only, there exists a threshold, $\delta^{\beta_0}$, such that for $\delta < \delta^{\beta_0}$, $M$ prefers to allow her consumers to have access to competitively supplied consumables of quality $\beta = 1$ than to lock them in to her own consumable. Conversely, for $\delta > \delta^{\beta_0}$, $M$ prefers to lock consumers in to her own consumable, denying them access to a competitively supplied consumable of equivalent quality. For $\delta = \delta^{\beta_0}$, $M$ is indifferent. The threshold, $\delta^{\beta_0}$, can be expressed as:

$$\delta^{\beta_0} = \begin{cases} \frac{1}{2} \left( -\rho + \sqrt{\rho(1+\rho)} \right) & \text{for } \rho \leq \frac{4}{5} \\ \frac{8(1+\rho)}{2(2+\rho)} & \text{otherwise} \end{cases} \quad (3.15)$$

In Figure 1 we compare the profits earned by $M$ with $\beta = 0$ to those with $\beta = 1$ for values of $\delta$ ranging between $(0, 1)$ for the case in which $\rho = 1$. For purposes of comparison, we have also included the total discounted profit that $M$ would earn under the optimal mechanism design solution over two periods, denoting $\pi^{MD} = (1+\rho)MD$. It can be seen that when $\delta \to 0$, and consumers are homogeneous, $M$’s profits under $\beta = 1$ converge to those under mechanism design. For this limiting case, $M$ can extract consumers’ full surplus through the price of the durable, but only if she can convince them that they will have future access to consumables at marginal cost. Thus, in this limiting case, $M$ can replicate the mechanism design solution by allowing consumers access to a competitively supplied consumable that is a perfect substitute ($\beta = 1$) for her own.
However, as we have previously discussed, M can sell her durable only for the price that the marginal consumer is willing to pay. As consumers become increasingly heterogeneous, this begins to leave more and more surplus for the higher valuation consumers. In contrast, if M locks-in her consumers by denying them access to this substitute consumable, then she obtains an additional lever that she can use to extract some additional surplus from the higher value consumers, but she also introduces the hold-up issue. When consumers are identical, i.e. \( \delta \to 0 \), this additional lever is unnecessary, and because of the hold-up issue, M earns roughly 12% less than by providing her consumers access to the alternative consumable. However, as \( \delta \) increases, the ability to use the price of her consumable to extract rents from the high valuation consumers becomes more valuable, and for \( \delta > 0.2 \) (in this example), M is better off denying her consumers access to the competitively supplied consumable.

![Figure 1: Comparison of profit under mechanism design versus linear pricing of the consumable with \( \beta = 0 \) or \( \beta = 1 \).](image)

Recall that the discount factor that is used in Figure 1 is \( \rho = 1 \). Because lower values of \( \rho \) imply more discounting of future cash flows, as we decrease the value of \( \rho \), it is obvious that all three of the profit functions shown in Figure 1 would decrease. In addition, the point at which \( \pi^{B0} = \pi^{B1} \) would shift to the left. This shifting of the indifference point can be verified mathematically by differentiating the upper and lower branches of (3.15) with respect to \( \rho \), but it can also be explained intuitively: As we increase the rate at which future utility is discounted, the hold-up problem is less important relative to the flexibility that lock-in provides to extract different amounts of surplus from different consumer types. In the extreme of \( \rho = 0 \), i.e. a single period problem, we
have \( \delta^{01} = 0 \) so that M is indifferent between \( \beta = 0 \) and \( \beta = 1 \) only as \( \delta \to 0 \), and strictly prefers \( \beta = 0 \) whenever there is any heterogeneity among the consumers. If we allow for a substitute consumable that is of intermediate quality, then identifying the optimal quantity and price in period 1 becomes a bit trickier since we need to allow for whether the quantity is above or below \( \bar{Q} \), the threshold derived in Lemma 3.2 as well as whether or not the consumables price in period 1 will attract purchases from the marginal consumer. Although we do not have a closed form expression for the optimal solution to M’s problem when \( \beta \in (0, 1) \), it is easy to compute it numerically. Denote by \( \pi^\beta \) the optimal profit that M can earn when she sells durables only in period 1, and her consumers have access to a competitively supplied consumable of quality \( \beta \).

In Figure 2 we investigate how a competitively supplied consumable of intermediate quality affects the profit of M and the quantity of durables that she produces versus \( \beta = 0 \) (no alternative consumable) and \( \beta = 1 \) (perfect substitute). In the figure, we have taken the intermediate level of quality to be \( \beta = 0.75 \), and \( \rho = 1 \). Notice that, except for very low values of \( \delta \), M’s profits are not monotone in \( \beta \), the quality of the competitively supplied consumable. For sufficiently low levels of consumer heterogeneity, M continues to prefer \( \beta = 1 \) due to its ability to mitigate hold-up. However, she prefers \( \beta = 0.75 \) to \( \beta = 0 \) over the full range of \( \delta \). That is, regardless of the amount of heterogeneity among her consumers, M can always benefit from her consumers having access to a competitively supplied consumable that is of at least the intermediate level \( (\beta = 0.75) \) of quality. Of course, as the discount factor decreases, so does the significance of the hold-up issue. Eventually, at sufficiently low values of \( \rho \), M will weakly prefer to have \( \beta = 0 \) (no competition from a substitute consumable) for all levels of consumer heterogeneity. In the extreme case of \( \rho = 0 \), M faces a single period problem and obtains no benefit in return for any sacrifice.
of her power to price the consumable. It is also of interest to see how M’s output of durables is affected by the quality of the substitute consumable. Plot b) in Figure 2 corroborates Corollary 3.7: As $\beta$ increases, M has less flexibility to extract rent through the price of her consumable, and she compensates for this by reducing her output of durables.

It is worth pointing out that, although our assumption that M can commit to shutting down production of her durable after period 1 is critical to this analysis, the assumption that there are only two periods is not. Once there are $Q$ durables available to consumers, M’s pricing problem with respect to the consumable is the same in every period. Therefore, even if there are an arbitrary number, $T$, of periods, M will set her consumables price as shown in Lemma 3.2 in each of periods $2, ..., T$. Moreover, if we apply a discount factor of $\rho_t$ to each of these periods $t = 2, ..., T$, then by interpreting the discount factor, $\rho$, that appears through our analysis of period 1 as $\rho = \sum_{t=2}^{T} \rho_t$ then all of our results hold for an arbitrary number of periods. Of course, with this interpretation, we could well have $\rho > 1$. For values of $\rho > 1$, the hold-up issue becomes relatively more important, and we would expect that M would prefer to allow her consumers access to a substitute consumable at higher values of $\delta$. Indeed, as previously discussed, it is easy to confirm that the threshold, $\delta^{01}$, from Proposition 3.8 is increasing in $\rho$.

### 3.3 M cannot commit to shutdown production of the durable in period 2

So far, we have assumed that M can commit to shut down her production of the durable after the first period. While this has allowed us to highlight the role that competition in the consumables market can play in mitigating the hold-up problem with respect to consumers, it ignores the other, more famous, time inconsistency issue in which M has an incentive to sell her durable to lower valuation consumers over time. To demonstrate that our main insights are robust with respect to this assumption, let us now relax the assumption that M can commit to shut down production of her durable after the first period. As before, we will continue to assume that there are only two periods, noting that, as in the famous paper by Bulow (1982), it is not critical that there be only two periods, but it is critical that there be a finite number of periods.

For this relaxed version of the problem, the sequence of events is exactly as described in Section 3.2, with one exception: In period 2, instead of just determining a price, $p_2$, for the consumable, M also determines an additional quantity of durables to make available. Let $Q_t$ be the total quantity of durables that are available to consumers in period $t = 1, 2$, so that the additional durables produced in period 2 is $Q_2 - Q_1$. Let $\Pi_2(Q_1, Q_2, p_2)$ be M’s second period profit given that $Q_1$ durables were sold in period 1, and that M sells $Q_2 - Q_1$ durables and sets the consumables price to $p_2$ in period. We can express these profits as:

$$\Pi_2(Q_1, Q_2, p_2) = (Q_2 - Q_1)r(\beta, Q_2, p_2) + p_2y(Q_2, p_2) \tag{3.16}$$

where we have used uppercase $\Pi$ to distinguish these profits from the ones where M can commit to shut down production of her durable after period 1. Denote the optimal solution to this second
period problem by \((Q_1^*, p_2^*)\). As in Section 3.2, this optimal solution to the period 2 problem is conditional upon the value of \(Q_1\). However, in contrast to the expression for \(\pi_2(Q, p_2)\) that appears in (3.8), M’s second period profit function now includes a term for the income from additional durables sales. In period 1, M’s profit function is:

\[
\Pi_1(\beta, Q_1, p_1) = Q_1(r(\beta, Q_1, p_1) + \rho r(\beta, Q_2^*, p_2^*)) + p_1y(Q_1, p_1) + \rho \Pi_2(Q_2^*, p_2^*)
\]

(3.17)

Because M’s problem in period 2 is not jointly concave in \(p_2\) and \(Q_2\), we have been unable to obtain a closed form solution to this problem. However, we are able to characterize the optimal consumable price in each period conditional upon the durables quantities. In a slight variation in the notation used in Section 3.2, we will denote by \(a_m(Q_t) = 1 + \delta (1 - Q_t)\) the type of the marginal consumer who holds a durable in period \(t\).

**Lemma 3.9.** In period 2, given \(Q_1\) and \(Q_2\), the conditionally optimal price of the consumable can be characterized as follows: If \(\Pi_2(Q_1, Q_2, p_h(Q_1, Q_2)) \geq \Pi_2(Q_1, Q_2, p_l(Q_1, Q_2))\), then \(p_2^o(Q_1, Q_2) = p_h(Q_1, Q_2)\), and otherwise, \(p_2^o(Q_1, Q_2) = p_l(Q_1, Q_2)\), where:

\[
p_h(Q_{t-1}, Q_t) = \max \left\{ \frac{(1 - \beta)(1 + \delta)}{2(1 - \beta) + \sqrt{1 - 2\beta + 4\beta^2}}, (1 - \beta) a_m(Q_t) \right\}
\]

(3.18)

\[
p_l(Q_{t-1}, Q_t) = \min \left\{ \frac{Q_{t-1} + (Q_{t-1} - 2Q_t Q_t + Q_t^2) \delta}{Q_{t-1} + Q_t}, (1 - \beta) a_m(Q_t) \right\}
\]

(3.19)

Because the second period decisions and profit are independent of the consumables price in period 1, the conditionally optimal consumables price, \(p_1^o(Q_1)\), can be characterized as follows: If \(\Pi_2(0, Q_1, p_h(0, Q_1)) \geq \Pi_2(0, Q_1, p_l(0, Q_1))\), then \(p_1^o(Q_1) = p_h(0, Q_1)\), and otherwise, \(p_1^o(0, Q_1) = p_l(0, Q_1)\).

The above Lemma can be explained as follows: In period 2, given \(Q_1\) and \(Q_2\), M can either price the consumable low enough to attract purchases from the marginal consumer, i.e. \(p_2 \leq (1 - \beta) a_m(Q_2)\), or she sets it above this threshold. When she sets the consumables price below this threshold, the marginal consumer purchases M’s consumable and the implicit rental price is \(r(1, Q_2, p_2)\). Otherwise, if M sets the consumables price above the threshold, then the marginal consumer prefers to purchase the competitively supplied consumable, and the implicit rental price is \(r(\beta, Q_2, 0)\). The first term inside the brackets in each of (3.18) and (3.19) represents the first order condition for each of these two constrained optimization problems, both of which are concave in \(p_2\). In period 1, we note that while \(p_1\) affects the implicit rental price for period 1, it has no impact upon the implicit rental price or the profits in period 2. Thus, for any \(Q_1\), M’s pricing problem in period 1 is exactly the same as the one that she would face in period 2 if \(Q_1' = 0\) and \(Q_2 = Q_1\).

Using this structure for M’s consumable pricing problem, we can use a search procedure to identify the optimal value of \(Q_2\) conditional upon \(Q_1\), and then subsequently search over the possible values for \(Q_1\). In Figure 3, we show two plots. In the plot a), we show how M’s profits vary with consumer heterogeneity (\(\delta\)) with \(\beta = 0, \beta = 0.75\), and \(\beta = 1\), the same quality parameters
that we used in Figure 2. Just as when M could commit to shutting down her output of durables in period 2, we see that $\Pi^{\beta_1}$ still dominates when $\delta \to 0$, but that as $\delta$ increases, $M$ would earn higher profit with either $\beta = 0.75$ or $\beta = 0$. However, in contrast to what we saw for the case in which $M$ could commit to shutting down durables production after the first period, we now see that, beyond about $\delta = 0.35$, $M$ prefers $\beta = 0$ (no competition from an alternative consumable) over $\beta = 0.75$. Moreover, for the larger values of $\delta$, it appears that the gap between $M$’s profit with $\beta = 0$ versus with $\beta = 1$ is larger in Figure 3 than in Figure 2.

![Figure 3: Comparison of profit when M can produce durables in both periods.](image)

This is even more evident from plot b) in Figure 3, where we show $M$’s profits for $\beta = 0$ and $\beta = 1$ as fractions of the profits that she could earn under the mechanism design solution. It can be observed that for low values of $\delta$, the ability to produce durables in period 2 has no consequence. This is because, when consumers are sufficiently homogeneous, $M$ sells durables to all of them in period 1, leaving her with no temptation to sell more in period 2. However, for values of $\delta > 0.4$, we can see that the ability to produce durables in period 2 hurts her a lot when $\beta = 1$. Indeed, when $\beta = 1$ and $\delta > 0.4$, the ability to continue to produce durables decreases her profits from about 88% of the mechanism design profits to only about 80%. Because $\beta = 1$ implies that consumers have access to consumables at marginal cost, this reduction in profit is due entirely to the manufacturer’s incentive to sell more durables that was recognized by Coase (1972), Bulow (1982), and others. However, somewhat surprisingly, when $\beta = 0$, the ability to continue to produce has a much different effect. For this case, where she faces no competition from an alternative consumable, she earns a slightly higher fraction of the mechanism design profits when she can continue durables production.

To see why this is the case, it is helpful to see how the durables quantities and consumables prices change when $M$ obtains the ability to produce durables in period 2. In Figure 4, we show
the durables quantities with production capability in both periods ($Q_1$ and $Q_2$) and compare them to the output quantity ($Q$) when durables can be produced only in period 1. In plot a) we show the comparison for $\beta = 1$, and in plot b) we show it for $\beta = 0$.

![Figure 4: Impact of the ability to produce durables in period 2 upon the quantity of durables.](image)

In the plots in Figure 4, we can see that, for both $\beta = 0$ and $\beta = 1$, we have $Q_1 \leq Q \leq Q_2$, i.e. when M obtains the ability to produce in period 2, she produces fewer durables in period 1, but her total output of durables by the end of period 2 is larger than when she can produce only in period 1. However, although this ordering is independent of $\beta$, both $Q_1$ and $Q_2$ are weakly larger when $\beta = 0$ than when $\beta = 1$, which is consistent with our result in Corollary 3.7 in the sense that competition in the consumables market leads to a smaller quantity of durables. Regardless of whether M can produce in period 2, the competition in the consumables market interferes with her ability to use the sales of consumables to extract more surplus from higher valuation consumers. Consequently, as she faces stiffer competition, i.e. larger $\beta$, she becomes less willing to put durables in the hands of low valuation consumers. When $\beta = 1$, and M cannot earn anything from consumables sales, her ability to produce durables in period 2 has only one effect: It creates an unavoidable temptation to sell to lower valuation consumers, eroding consumers willingness to pay for the durable in period 1.

On the other hand, when $\beta = 0$, the story is quite different. For this case, M has a monopoly in the consumables market. Although the ability to produce durables in period 2 still creates an unavoidable temptation to sell to lower valuation consumers, it also affects M’s incentives with respect to the prices ($p_1$ and $p_2$) that she sets for the consumable.
As can be seen in Figure 5, for the values of $\delta$ for which $M$ does not sell to all consumers in period 1, i.e. $\delta > \frac{1}{2}$, the ability to produce durables in period 2 causes her to offer lower consumables prices in both period 1 and in period 2. In period 1, because $Q_1 \leq Q$, there is less heterogeneity among the consumers who hold the durable, and this causes a shift in $M$’s priorities toward increasing the size of the surplus that she can extract from the marginal consumer. By offering a lower $p_1$, she increases the size of the surplus that can be extracted through the price of the durable, but gives up some of the additional surplus that she could have extracted from the higher valuation consumers. However, the reduction in the amount of heterogeneity among the consumers holding the durable makes her willing to make this trade-off. In period 2, $M$’s ability to produce and sell more durables endows her with a mechanism for extracting the surplus from the marginal consumer in period 2 that she did not previously have. By lowering the price of her consumable, she can expand the magnitude of this extractable surplus. As a consequence of this effect that $M$’s ability to sell additional durables has upon her incentives for pricing the consumable, the ability to continue to sell durables is less of a problem for her when she can lock her consumers in to a proprietary consumable. In fact, as shown in Figure 3, $M$’s ability to produce durables in period 2 actually increases her profit when she monopolizes the consumables market with $\beta = 0$, which contrasts sharply with existing results for durable goods manufacturers that are based on an implicit assumption that $\beta = 1$, i.e. consumers have free access to any consumables that might be needed.

4 Summary and Discussion

We have examined the choice that many manufacturers of durable products face when their products cannot be used without a contingent consumable. Such a manufacturer ($M$) must choose
between locking-in consumers by making her durable incompatible with consumables other than her own versus welcoming competition from substitute consumables. In order to examine this question, we have developed a micro model of how consumers derive utility from a durable product in each period for which they have access to it. Our model is based on the idea that consumers derive decreasing marginal utility from each use of the durable, and that consumers may vary from one another in their maximum marginal utilities.

We first derive the solution to a mechanism design problem in which the firm can provide consumers with a continuous menu of consumables quantities and prices in each period. Such an approach clearly represents an upper bound on M’s profits, and if it were implementable, then M would have no reason to do anything other than lock-in her consumers. However, there are reasons why the mechanism design solution may not be implementable, and we see many real examples of manufacturers forgoing approaches that resemble mechanism design solutions in favor of selling their durables and offering consumables at constant per-unit prices. It is our hope that our analysis will help to inform such manufacturers about the extent to which they should welcome or discourage competition in the consumables market.

In order to highlight the hold-up issue that arises with respect to M’s incentives with respect to her consumables price, we first consider a situation in which M can produce durables only in the first period, but consumers anticipate using it beyond the period in which they purchase. For this simplest setting, we demonstrate how a lock-in policy is something of a double edged sword. On the one hand, it endows M with an additional lever for extracting consumer surplus. In addition to the price of the durable, which extracts the full surplus from the marginal consumer, the price of M’s consumable provides a means of extracting additional surplus from the higher valuation consumers. On the other hand, this pricing power in the consumables market also creates a hold-up problem in which consumers’ willingness to pay for the durable is adversely affected by M’s unavoidable temptation to set higher consumables prices after she sells some durables. By allowing consumers access to competitively supplied consumables, M sacrifices pricing power in the consumables market, which mitigates the hold-up problem but also coarsens her means of surplus extraction. Because of this trade-off, we find that when consumers are sufficiently homogeneous, M should welcome competition from alternative consumables of comparable quality to her own. But when consumers become more heterogeneous, M derives more value from a relatively refined surplus extraction mechanism that lock-in affords her than she does from mitigating the hold-up problem. Consequently, when consumers vary substantially from one another in their marginal valuations, M prefers a lock-in strategy that provides her with at least some pricing power in the consumables market. Yet in these cases of large amounts of consumer heterogeneity, where M is better off with no competition than with perfect competition, she may be even better off with imperfect competition from a lower quality consumable. This occurs as a consequence of the fact that competition from a lower quality consumable allows M to credibly commit to lower future consumables prices without completely giving up her ability to extract additional surplus from the higher valuation consumers.
We then extend our analysis to the case in which M can produce durables in each of two periods. Through a numerical study we see that the main qualitative insights carry over from our earlier analysis. Specifically, M still prefers perfect competition in the consumables market when consumers are homogeneous, and she prefers to lock-in her consumers when the amount of heterogeneity among consumers exceeds a given threshold. However, we also see that M’s ability to produce (or inability to commit not to produce) in period 2 has dramatically different effects depending upon whether M faces perfect competition or no competition (lock-in) in the consumables market. In the former case, M’s profits come entirely from sales of durables, and her ability to produce in period 2 only causes consumers to anticipate lower durables prices which reduces their willingness to pay in period 1. However, under lock-in, M’s ability to produce durables in period 2 also affects her incentives with respect to the price of the consumable, causing her to offer a lower price in both periods. As a result, when M has pricing power in the consumables market, the ability to produce additional durables in period 2 need not be a handicap as it is for a durable goods monopolist whose product either does not require a consumable or is compatible with a competitively supplied consumable.

It is worth noting that, as is the case for many of the models that are used to examine intertemporal issues related to product durability, M’s incentive to sell durables in each period depends upon the distribution of valuations among consumers who have yet to purchase it. In addition, when M controls the price of the consumable, her incentives depend not only upon the distribution among consumers who have yet to purchase, but also upon the distribution of valuations among consumers who have already purchased. Because both of these distributions change after every period in which M sells a positive quantity of durables, the basic dynamics of how M’s incentives change over time do not depend on there being only two periods, and our qualitative insights extend to any finite time horizon in which the total size of the market and the distribution of valuations of consumers within it remains constant.

Throughout our analysis, we have assumed that consumables must be consumed in the period in which they are produced. This is easily justified in contexts for which consumables are perishable, e.g. cell-phone minutes, or where the content may change over time, e.g. the e-books that consumers will want next year have yet to be written. On the other hand, there are also situations in which consumers could stockpile consumables for future consumption. For example, there would be little to prevent consumers, or arbitragers, from stockpiling ink-cartridges in anticipation of higher future prices. When this is the case, stockpiling can at least partially mitigate the hold-up problem, but it also introduces some interesting new trade-offs. Recall that when M relies on competition in the consumables market to mitigate hold-up, she sacrifices pricing power as well as perhaps some unit sales. In contrast, with stockpiling, M produces consumables in period 1 that are not used until period 2, and this holding cost is a dead-weight loss. In addition, when hold-up is mitigated by stockpiling instead of by competition, it is likely to have different effects upon M’s incentives with respect to durables production. Because of these important distinctions between stockpiling and competition as means of mitigating M’s hold-up problem, stockpiling is
a worthy subject for future research. Of course, another potentially interesting direction for future research would be to consider the role that competition from another durable goods manufacturer may play in M’s decision about whether to lock-in consumers.

References


Appendix

**Proof of Lemma 3.1**

We apply the standard mechanism design results based on Maskin and Riley (1984)'s derivation for the continuous distribution model to derive the optimal quantity-price pair. Using the information rent received by a consumer of type \( a \) is \( V(a) = U(a, q(a)) - t(a) \), i.e. the difference between his total utility and the tariff that he pays, we can write the objective function of problem (MD) as follows:

\[
\max_{t(a), q(a)} \int_{1-\delta}^{1+\delta} (U(a, q(a)) - V(a)) f(a) da = \max_{q(a)} \int_{1-\delta}^{1+\delta} \left( U(a, q(a)) - \int_{1-\delta}^a q(\tau) d\tau \right) f(a) da \quad (A.1)
\]

where the expression on the right-hand side comes from substituting the incentive compatibility constraint (3.3). We can now use an integration by parts to express the above objective function as:

\[
\max_{q(a)} \int_{1-\delta}^{1+\delta} \left( U(a, q(a)) - \frac{1 - F(a)}{f(a)} \right) f(a) da \quad (A.2)
\]

recognizing that \( U(a, 1, q(a)) = q(a)(2 - q(a))/2 \) and that \( f(a) = \frac{1}{2\delta} \) and \( F(a) = \frac{a + \delta - 1}{2\delta} \) for a \( U(1-\delta, 1+\delta) \) distribution, we can use the first-order condition to maximize pointwise to obtain, \( q(a)^* = \max \{0, 2a - (1+\delta)\} \). Because the lowest valuation consumer is type \( a = 1 - \delta \), it is easy to confirm that all consumer types \( a \in (1 - \delta, 1 + \delta) \) will be allocated positive quantities so long as \( \delta < \frac{1}{3} \). For these low values of \( \delta \), we obtain the total price paid by a consumer of type \( a \) as:

\[
t(a) = U(a, q(a)) - \int_{1-\delta}^a (2\tau - (1 + \delta)) d\tau = \frac{4a(1+\delta) - 2a^2 - 3(2 - \delta)\delta - 1}{2} \quad (A.3)
\]

When \( \delta \geq \frac{1}{3} \), then only those consumers of type \( a > \frac{1+\delta}{2} \) receive \( q(a) > 0 \). For these consumers \( a \in (\frac{1+\delta}{2}, 1 + \delta) \), the total price paid can be obtained as:

\[
t(a) = U(a, q(a)) - \int_{(1+\delta)/2}^a (2\tau - (1 + \delta)) d\tau = \frac{8a(1+\delta) - 4a^2 - 3(1 + \delta)^2}{4} \quad (A.4)
\]

The expressions for the profit associated with the optimal solution to problem MD can be obtained by integrating (A.3) or (A.4) over the appropriate range of \( a \) depending on whether \( \delta \geq \frac{1}{3} \).

■

**Proof of Lemma 3.2**
By substituting (3.7) into (3.8), the second period profits can be expressed as:

\[
\pi_2(Q, p_2) = \begin{cases} 
Q(1 - p_2 + \delta(1 - Q))p_2 & \text{for } p_2 \leq (1 - \beta)a_m \\
\frac{(p_2 - (1 - \beta)(1 + \delta))(p_2(1 - 2\beta) - (1 - \beta)(1 + \delta))}{4(1 - \beta)^2}p_2 & \text{otherwise}
\end{cases}
\]  
(A.5)

The lower branch is unimodal in \( p_2 \) with a maximum at \( p_2^l = \frac{(1 - \beta)(1 + \delta)}{2(1 - \beta) + \sqrt{1 - 2\beta + 4\beta^2}} \), to see this, we look at the first derivative. Ignoring the constant term, \( \left( 4(1 - \beta)^2 \delta \right)^{-1} \), this derivative can be expressed as:

\[
p_2^2(3 - 6\beta) - 4p_2 (1 - \beta)^2 (1 + \delta) + (1 - \beta)^2 (1 + \delta)^2
\]

The first derivative is continuous and the roots of the above expression are: \( i_1 = \frac{(1 - \beta)(1 + \delta)}{2(1 - \beta) + \sqrt{1 - 2\beta + 4\beta^2}} \) and \( i_2 = \frac{(1 - \beta)(1 + \delta)}{2(1 - \beta) - \sqrt{1 - 2\beta + 4\beta^2}} \). The root \( i_2 \) is either negative or greater than \((1 - \beta)(1 + \delta)\). Note that \((1 - \beta)(1 + \delta)\) is the maximum price that the manufacturer can charge for the consumable so as to induce purchase of her consumable. Therefore \( i_2 \) is not a feasible price for the consumable. Given that \( i_2 < 0 < i_1 \) or \( 0 < i_1 < i_2 \) we can see that the first derivative is positive when \( p \in (0, i_1) \) and negative for \( p \in (i_1, (1 - \beta)(1 + \delta)] \). Therefore the lower branch of A.5 is unimodal at \( i_1 \). Further \( i_1 = p_2^l > (1 - \beta)a_m \) if and only if \( Q > \tilde{Q} = \frac{\beta(1 + \delta)}{\delta(4\beta - 1 + \sqrt{1 - 2\beta + 4\beta^2})} \).

It follows that if \( Q \leq \tilde{Q} \), the optimal price will be \( p_2^l \leq (1 - \beta)a_m \). It is easy to confirm that the upper branch of (A.5) is concave in \( p_2 \), and that its first order condition is satisfied at: \( p_2^u = \frac{(1 + \delta)(1 - Q)}{\delta(4\beta - 1 + \sqrt{1 - 2\beta + 4\beta^2})} \). It follows that the optimal price is: \( \text{Min} \{ (1 - \beta)a_m, p_2^u \} \).

**Proof of Corollary 3.3**

The partial derivatives of \( \tilde{Q} \) with respect to \( \beta \) and \( \delta \) are as follows:

\[
\frac{\partial \tilde{Q}}{\partial \beta} = -\frac{(1 + \delta) \left( -1 + \beta + \sqrt{1 - 2\beta + 4\beta^2} \right)}{\delta \left( 1 - 4\beta + \sqrt{1 - 2\beta + 4\beta^2} \right)^2 \sqrt{1 - 2\beta + 4\beta^2}}
\]

\[
\frac{\partial \tilde{Q}}{\partial \delta} = -\frac{\beta}{\delta^2 \left( 1 - 4\beta + \sqrt{1 - 2\beta + 4\beta^2} \right)}
\]

By differentiating twice with respect to \( \beta \), we can confirm that the term, \(-1 + \beta + \sqrt{1 - 2\beta + 4\beta^2}\), is convex, and that it is minimized at \( \beta = 0 \). Evaluating this term for \( \beta = 0 \), confirms that \(-1 + \beta + \sqrt{1 - 2\beta + 4\beta^2} \geq 0 \). It follows immediately that \( \frac{\partial \tilde{Q}}{\partial \delta} < 0 \). Since \( 1 - 2\beta + 4\beta^2 > 0 \) for all \( \beta \in [0, 1] \), so we also have that \( \frac{\partial \tilde{Q}}{\partial \beta} < 0 \).

We can see that \((1 - \beta)a_m\) is non-increasing in \( \beta \). Therefore the \( \text{Min} \left\{ (1 - \beta)a_m, \frac{(1 + \delta)(1 - Q)}{2} \right\} \) is non-increasing in \( \beta \). Similarly by evaluating the expression \( \frac{\partial p_2^l}{\partial \beta} \) we obtain the following expression:
which is clearly non-positive for all $\beta \in [0,1]$.

To show how $p^*_2$ responds to changes in $\delta$, we first observe that both $p^*_2 = \frac{(1-\beta)(1+\delta)}{2(1-\beta)+\sqrt{1-2\beta+4\beta^2}}$ and $p^*_2 = \frac{(1+\delta(1-Q))}{2} \frac{1}{\delta}$ are increasing in $\delta$, while $(1-\beta)a_m = (1-\beta)(1+\delta(1-2Q))$ is increasing (decreasing) in $\delta$ for $Q < \frac{1}{2}$ ($Q > \frac{1}{2}$). Thus, from the definition of $p^*_2$ in Lemma 3.2, we can see that it is decreasing in $\delta$ if and only if $(1-\beta)a_m < p^*_2$, $Q > \frac{1}{2}$, and $Q < Q$. The result follows from the fact that $(1-\beta)a_m > p^*_2$ if and only if $Q < \frac{(2\beta-1)(1+\delta)}{(4\beta-1)a}$. ■

**Proof of Lemma 3.4**

By substituting (3.9) into (3.10), we obtain the following expression for the price of the durable in period 1:

$$p_{d1}(\beta) = \begin{cases} \frac{(a_m-p_1)^2}{2} + \rho r(\beta, Q, p^*_2) & \text{for } p_1 \leq a_m/(1-\beta) \\ \frac{a_m^2}{2} + \rho r(\beta, Q, p^*_2) & \text{for } p_1 \geq a_m/(1-\beta) \end{cases}$$

where $a_m = 1 + \delta(1-2Q)$ and $r(\beta, Q, p^*_2)$ can be obtained by substituting $p^*_2$ from Lemma 3.2 into (3.9). Differentiating with respect to $\beta$, we have:

$$\frac{dp_{d1}(\beta)}{d\beta} = \begin{cases} \rho \left( \frac{\partial r(\beta, Q, p^*_2)}{\partial \beta} + \frac{\partial r(\beta, Q, p^*_2)}{\partial p^*_2} \frac{\partial p^*_2}{\partial \beta} \right) & \text{for } p_1 \leq a_m/(1-\beta) \\ a_m + \rho \left( \frac{\partial r(\beta, Q, p^*_2)}{\partial \beta} + \frac{\partial r(\beta, Q, p^*_2)}{\partial p^*_2} \frac{\partial p^*_2}{\partial \beta} \right) \frac{\partial p^*_2}{\partial \beta} & \text{for } p_1 \geq a_m/(1-\beta) \end{cases}$$

(A.6)

It is obvious that $a_m \beta$ is increasing in $\beta$, so we will focus out attention upon the term that is common to both the upper and lower branches of (A.6). From (3.9) we can see that if $p^*_2 \leq a_m/(1-\beta)$, then $\frac{\partial r(\beta, Q, p^*_2)}{\partial \beta} = 0$ while $\frac{\partial r(\beta, Q, p^*_2)}{\partial p^*_2} < 0$. On the other hand, if $p^*_2 \geq a_m/(1-\beta)$, then $\frac{\partial r(\beta, Q, p^*_2)}{\partial \beta} = a_m^2 \beta > 0$ while $\frac{\partial r(\beta, Q, p^*_2)}{\partial p^*_2} = 0$. The result follows from the fact that $\frac{\partial p^*_2}{\partial \beta} \leq 0$, which was established in Corollary 3.3. ■

**Proof of Lemma 3.5**

For the case in which $\beta = 1$, M’s profit is $\pi_1(1, Q, 0)$ as shown in (3.12). Differentiating with respect to $Q$, we have:

$$\frac{d\pi_1(1, Q, 0)}{dQ} = \frac{1 + \rho}{2} \left( 1 + \delta(1-2Q) \right) (1 + \delta (1-6Q))$$

(A.7)

which has two roots: $j_1 = \frac{1 + \delta}{6\delta}$ and $j_2 = \frac{1 + \delta}{2\delta}$. $j_2$ is clearly greater than 1 for all $\delta < 1$. It can be easily verified that $\frac{d\pi_1(1, Q, 0)}{dQ} > 0$ for $Q \in [0, j_1)$ and $\frac{d\pi_1(1, Q, 0)}{dQ} < 0$ for $Q \in (j_1, 1]$. Therefore
\( \pi_1(1, Q, 0) \) is unimodal in \( Q \), with a maximum at \( j_1 \). Further, \( j_1 \leq 1 \) only if \( \delta \geq \frac{1}{5} \). Therefore the optimal quantity is given by

\[
Q^{\beta_1} = \begin{cases} 
1 & \text{for } \delta \leq 1/5 \\
\frac{1 + \delta}{6\delta} & \text{for } \delta \geq 1/5
\end{cases}
\]

The expressions for the optimal profits can be obtained by substituting \( Q^{\beta_1} \) into (3.12).

**Proof of Lemma 3.6**

When \( \beta = 0 \) M’s profits in period 1 can be obtained by substituting which can be obtained by substituting (3.14) into (3.11) to obtain:

\[
\pi_1(0, Q, p_1) = \begin{cases} 
\frac{Q}{2} \left( -p_1^2 + 2p_1Q\delta + (1 + (2-2Q)\delta)^2 \right) + \frac{Q^3}{8}(3 + \delta (11Q^2\delta - 10Q(1 + \delta) + 3(2 + \delta))) & \text{for } Q \leq (1 + \delta)/3\delta \\
\frac{1}{5\delta} (27Q\delta (-p_1^2 + 2p_1Q\delta + (1 + (2-2Q)\delta)^2) + 2(1 + \delta)^3\rho) & \text{otherwise}
\end{cases}
\]

(A.8)

It is easy to confirm that \( \frac{d\pi_1(0, Q, p_1)}{dp_1^2} < 0 \) so that M’s profits are concave in \( p_1 \). The FOC for both the upper and lower branches of (A.8) is \( p_1 = Q\delta \), so that the conditionally optimal price for any \( Q \) is \( p_1^{co}(Q) = \operatorname{Min}\{\delta Q, a_m\} \), which ensures that the marginal consumer is at least indifferent to using the product. We now note that, \( p_1^{co}(Q) = \delta Q \) if and only if \( Q \leq \frac{1+\delta}{3\delta} \). Thus, when we substitute \( p_1^{co}(Q) = \delta Q \) and \( p_1^{co}(Q) = a_m \) into the upper and lower branches of (A.8) and differentiate with respect to \( Q \), we obtain:

\[
\frac{d\pi_1(0, Q, Q\delta)}{dQ} = \begin{cases} 
\frac{1}{8}(1 + (1 - 3Q)\delta) (4 + 3p + \delta (4 + 3p - Q(20 + 11\rho))) & \text{for } Q \leq (1 + \delta)/3\delta \\
2Q\delta(1 + \delta(1 - 3Q)) & \text{otherwise}
\end{cases}
\]

(A.9)

It is easy to confirm that the lower branch is negative for all \( Q > \frac{1+\delta}{3\delta} \), so we will never set the durables quantity above this threshold. The lower branch of (A.9) has two roots, \( k_1 = \frac{(1 + \delta)(4 + 3p)}{\delta(20 + 11\rho)} \) and \( k_2 = \frac{1+\delta}{3\delta} \). It can be confirmed that \( k_1 < k_2 \) and that \( \frac{d\pi_1(0, Q, Q\delta)}{dQ} > 0 \) for \( Q < k_1 \) and negative beyond \( k_1 \). Therefore the optimal quantity of durables can be represented as \( Q^{\beta_0} = \operatorname{Min}\{k_1, 1\} \), where \( k_1 < 1 \) only if \( \delta > \frac{4+3p}{8(2+\rho)} \). By substituting \( Q^{\beta_0} \) and \( p_1^{co}(Q^{\beta_0}) \) into (A.9), we can confirm that the optimal profits for \( \beta = 0 \) are as stated in the Lemma.

**Proof of Corollary 3.7**
Note that \( \frac{4+3\rho}{8(2+\rho)} > \frac{1}{5} \) for all \( \rho \in [0, 1] \). For \( \delta < \frac{1}{5} \), \( Q^{\beta_0} = 1 = Q^{\beta_1} \). When \( \delta \in \left[ \frac{1}{5}, \frac{4+3\rho}{8(2+\rho)} \right] \) then \( Q^{\beta_0} = 1 > \frac{(1+\delta)^{10}}{8} = Q^{\beta_1} \). For \( \delta > \frac{4+3\rho}{8(2+\rho)} \), \( Q^{\beta_0} = \frac{(1+\delta)(4+3\rho)}{20(2+11\rho)} > \frac{(1+\delta)}{8} = Q^{\beta_1} \) so long as \( \frac{4+3\rho}{20(2+11\rho)} > \frac{1}{5} \) which is always true.

**Proof of Proposition 3.8**

By substituting (3.9) into (3.16), M’s problem in period 2 reduces to maximizing the following:

\[
\Pi_2(Q_{t-1}, Q_t, p_t) = \begin{cases} 
(1 - p_2 + \delta(1 - Q_t))p_t + \frac{(Q_t - Q_{t-1})}{2} (a_m(Q_t) - p_t)^2 & \text{for } p_t \leq (1 - \beta)a_m(Q_t) \\
(p_2(\beta(1+\delta))p_1(1-2\beta) - (1-\beta)(1+\delta) \frac{4(1-\beta)^2}{\delta})p_t + \frac{(Q_t - Q_{t-1})^2}{2} a_m(Q_t)^2 & \text{otherwise}
\end{cases}
\]

(A.10)
It is easy to see that the upper branch of (A.10) is concave in $p_t$, for any $0 \le Q_1 \le Q_2 \le 1$. As established in the proof of Lemma 3.2, the lower branch is unimodal in $p_t$. Therefore the FOC to the upper and lower branches are:

$$\text{FOC}^u_p = \frac{Q_{t-1} + (Q_{t-1} - 2Q_{t-1}Q_t + Q_t^2)\delta}{Q_{t-1} + Q_t} \quad \text{and} \quad \text{FOC}^l_p = \frac{(1 - \beta)(1 + \delta)}{2(1 - \beta) + \sqrt{1 - 2\beta + 4\beta^2}}$$

respectively. From the limit on the prices of $p_t \le (1 - \beta)a_m$ on the upper branch and $p_t \ge (1 - \beta)a_m$ on the lower branch the conditionally optimal prices given $Q_1$ and $Q_2$ and either the constraint that $p_t \le (1 - \beta)a_m(Q_t)$ or $p_t \ge (1 - \beta)a_m(Q_t)$:

$$p_l(Q_1, Q_2) = \text{Min}\{\text{FOC}^u_{p_2}, (1 - \beta)a_m\} \quad \text{and} \quad p_h(Q_1, Q_2) = \text{Max}\{\text{FOC}^l_{p_2}, (1 - \beta)a_m\}$$

respectively, and it is easy to see that the manufacturer chooses the optimal price of consumable $p_2^o(Q_1, Q_2)$ from $p_l$ and $p_h$, whichever results in a higher profit.

By substituting the optimal second period quantity of durables, $Q_2^*$ and $p_2^*$ into (3.17), the manufacturer’s problem in period 1 reduces to maximizing the following profit function:

$$\Pi_1(\beta, Q_1, p_1) = \begin{cases} 
(1 - p_1 + \delta(1 - Q_1))p_1 + \frac{Q_1}{2} (a_m(Q_1) - p_1)^2 
+ \rho \Pi_2(Q_1, Q_2^*, p_2^*) & \text{for } p_1 \le (1 - \beta)a_m(Q_1) \\
\frac{(p_1 - (1 - \beta)(1 + \delta))p_1(1 - 2\beta) - (1 - \beta)(1 + \delta)}{4(1 - \beta)^2\delta}p_1 + \frac{Q_1^2}{2}a_m(Q_1)^2 
+ \rho \Pi_2(Q_1, Q_2^*, p_2^*) & \text{otherwise}
\end{cases}$$

(A.11)

Note that $Q_2^*$ and $p_2^*$ are a function of $Q_1$ but independent of $p_1$. Given this, for the purposes of identifying the value of $p_1$ that maximizes (A.11) conditional upon $Q_1$, we can ignore the term, $\rho\Pi_2(Q_1, Q_2^*, p_2^*)$, that appears in both the upper and lower branches. After ignoring these terms, we can compare (A.11) to (A.10) and it is easy to see that the value of $p_t$ that maximizes (A.10) for $Q_{t-1} = 0$ and $Q_t = Q_1$ also maximizes (A.11) for a given value of $Q_1$. ■